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INVARIANT ENERGY INTEGRALS FOR THE NON-LINEAR CRACK PROBLEM WITH POSSIBLE CONTACT OF THE CRACK SURFACES†

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Generally two-dimensional and three-dimensional formulations of the non-linear crack problem when the crack surfaces do not overlap for a non-uniform anisotropic linearly elastic body are considered. The first derivative of the potential energy function with respect to the perturbation parameter and its representation in the form of an invariant integral over an arbitrary closed contour are obtained for a general form of the differentiable perturbation of a region with a cut, using the method of material derivatives. The sufficient conditions for the existence of an invariant energy integral are derived in general form, and examples of invariant integrals are constructed for different types of perturbations and a different geometry of the cut.

We consider the non-linear problem of the equilibrium of body containing a crack, when the condition for the crack surfaces not to overlap is satisfied. A general two-dimensional or three-dimensional model of a linearly elastic, inhomogeneous, anisotropic body is used. Unlike the classical linear formulation of the crack problem under conditions when the crack surfaces are stress-free, the non-linear problem is reduced to a variational inequality. The perturbations of this problem are investigated in order to obtain a general form of the invariant energy integrals. A classical example of invariant energy integrals is the Cherepanov–Rice integral, which describes the energy-release rate (equal to the rate of inflow of energy at the crack tip) and is used in fracture mechanics to describe the growth of the crack. The path independent integral was defined in [1–3] in the essentially non-linear problems of the non-linearly elastic and inelastic deformations of materials with cracks. Some crack problems were investigated in [4, 5] when there was unilateral contact between the crack surfaces. A mathematical basis for the invariant energy integrals for linear problems was presented in [6, 7].

The theory of singular perturbations [8, 9] is usually employed when considering regions with non-smooth boundaries, for example, bodies with cracks. Regular perturbations, despite the non-smoothness of the boundary, are used when representing a region by means of smooth coordinate transformations. This approach extends shape-optimization methods for smooth regions [10] to regions with non-smooth boundaries. The derivative of the energy in the non-linear problem was obtained for the first time in [11]. Using the variational formulation of linear and non-linear crack problems [12, 13], formulae were then obtained for the derivative of the potential energy and the corresponding invariant integrals for a number of cases [14, 15].

In Section 1 we investigate the problem of the general perturbation of a region with a cut, where the cut geometry is not specified. The initial problem of the equilibrium of a body with a crack is formulated in the form of a variational inequality; the perturbed problem in a region with a crack is constructed, and the conditions for the mutual uniqueness of its representation in the initial region are formulated; the asymptotic form of the perturbed solution is derived and the material derivative of the solution is determined; using the material derivative the asymptotic form of the potential energy function and a formula for its first derivative are obtained.

Section 2 is devoted to the use of the relations obtained in Section 1 to find the invariant integrals. A general form of the invariant energy integral and the sufficient conditions for its existence are obtained; using the general formula, examples of invariant integrals are constructed using perturbations of the whole cut, the edge of the cut and when the tip of the cut is perturbed, which will also hold for the classical linear crack problem.

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1. THE GENERAL PERTURBATION OF A REGION WITH A CUT

Formulation of the initial problem. Consider a bounded region $\Omega \subset \mathbf{R}^N$, where $N = 2$ or 3, with a Lipschitz-continuous boundary $\partial\Omega$ and a certain part of it $\Gamma_D \subseteq \partial\Omega$ with meas $\Gamma_D > 0$. Suppose there is a cut Γ_0 inside Ω like a certain $(N-1)$-dimensional manifold in $\mathbf{R}^N$, i.e. Γ_0 is a non-closed curve when $N = 2$ or a surface when $N = 3$. We will denote the edge of the cut by γ_0.

Assumption 1. The set $(\Omega, \Gamma_0, \Gamma_D)$ satisfies the following condition. The region Ω can be divided into subregions Ω_1 and Ω_2, i.e. $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$, with a common boundary Γ, i.e. $\overline{\Omega}_1 \cup \overline{\Omega}_2 = \overline{\Gamma}$, and in this case the following conditions are satisfied: (a) Ω_1 and Ω_2 have Lipschitz-continuous boundaries $\partial\Omega_1$ and $\partial\Omega_2$, (b) $\Gamma_0 \subset \Gamma$, and (c) meas $(\partial\Omega_n \cap \Gamma_D) > 0$ for $n = 1, 2$.

Conditions *a* and *b* define the smoothness of the region with the cut, while condition *c* is necessary in order to satisfy the Korn inequality. In particular, it follows from Assumption 1 that the cut Γ_0 is Lipschitz-continuous, and hence we can choose the unit vector $\nu = (\nu_1, \dots, \nu_N)$ normal to Γ_0 at least almost everywhere on the cut. We will assume that the chosen direction ν corresponds to the positive side of the cut Γ_0^+ while $-\nu$ corresponds to the negative side Γ_0^-. We will now define the region with the cut in $\mathbf{R}^N$ as $\Omega_0 = \Omega\backslash\overline{\Gamma}_0$ with the boundary $\partial\Omega_0 = \partial\Omega \cup \Gamma_0^+ \cup \Gamma_0^- \cup \gamma_0$.

Everywhere henceforth, unless otherwise stated, the subscripts i, j, k and l take the values $1, \dots, N$; summation is carried out over repeated subscripts.

We will introduce the Sobolev space

$$\tilde{H}^1(\Omega_0) = \{ \upsilon = (\upsilon_1, \dots, \upsilon_N)\colon \upsilon_i \in H^1(\Omega_0),\ \upsilon_i = 0 \text{ almost everywhere on } \Gamma_D \}$$

which contains the homogeneous Dirichlet condition (the fixing of the body) on part of the external boundary Γ_D. We will require that the condition for no overlapping of the sides of the cut to be satisfied; this can be written in the form of the following inequality for the jump in the function on the cut [12]

$$[\upsilon] \cdot \nu \equiv [\upsilon_i]\nu_i \geq 0, \quad [\upsilon_i] = \upsilon_i|_{\Gamma_0^+} - \upsilon_i|_{\Gamma_0^-}$$

This condition leads to the definition of the set of permissible functions in the form

$$K_0 = \{ \upsilon = (\upsilon_1, \dots, \upsilon_N) \in \tilde{H}^1(\Omega_0)\colon [\upsilon] \cdot \nu \geq 0 \text{ almost everywhere on } \Gamma_0 \}$$

which is convex and closed in $\tilde{H}^1(\Omega_0)$. Within the framework of the linear theory of elasticity, for the displacement vector $\upsilon = (\upsilon_1, \dots, \upsilon_N)$, we define, in a standard way, the deformation and stress tensors

$$\epsilon_{ij}(\upsilon) = \frac{1}{2}(\upsilon_{i,j} + \upsilon_{j,i}), \quad \sigma_{ij}(\upsilon) = c_{ijkl}\epsilon_{kl}(\upsilon)$$

with a symmetrical and positive-definite tensor of the coefficients of elasticity $\{c_{ijkl}\}$, i.e. $c_{ijkl} = c_{jikl} = c_{klij}$ and $c_{ijkl}\xi_{kl}\xi_{ij} \geqslant c_0\xi_{ij}\xi_{ij} > 0$. Here we assume that $c_{ijkl} \in C^2(\mathbf{R}^N)$.

Suppose the external load in the region $f = (f_1, \dots, f_N)$ is also specified by smooth functions $f_i \in C^2(\mathbf{R}^N)$. The following problem of the theory of elasticity is considered in the region with a cut Ω_0 when the sides of the cut Γ_0 do not overlap in the generalized formulation

$$\int \sigma_{ij}(u^0)\epsilon_{ij}(\upsilon - u^0)d\Omega_0 \geq \int f_i(\upsilon - u^0)_i d\Omega_0, \quad \forall \upsilon \in K_0 \tag{1.1}$$

By virtue of Assumption 1 and the previous constructions, a unique solution $u^0 \in K_0$ of variational inequality (1.1) exists. It is characterized by the following relations

$$\begin{gathered} -\sigma_{ij,j}(u^0) = f_i \quad \text{almost everywhere in} \quad \Omega_0 \\ [u^0] \cdot \nu \geq 0, \quad [\sigma_{ij}(u^0)\nu_j] = 0, \quad \sigma_{kj}(u^0)\nu_j\nu_k \leq 0 \\ \sigma_{ij}(u^0)\nu_j - \sigma_{kj}(u^0)\nu_j\nu_k\nu_i = 0, \quad \sigma_{kj}(u^0)\nu_j\nu_k([u^0] \cdot \nu) = 0 \quad \text{on} \quad \Gamma_0 \end{gathered} \tag{1.2}$$

The relations on the cut can be given an exact meaning in the space $H_{00}^{1/2}(\Gamma_0)$ for a jump in the displacements and its dual space with the corresponding stresses, as was demonstrated previously in [12], if we additionally require the $C^{1,1}$-smoothness of the cut. Moreover, it was shown in [12] that the solution u^0 of problem (1.1) possesses an additional local H^2-smoothness inside the region Ω_0 and up to the sides of the cut $\Gamma_0^{\pm}$ excluding the neighbourhood of the edge of the cut γ_0.

Formulation of the perturbed problem. For the small parameter $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ we take the perturbation $\Phi = (\Phi_1, \ldots, \Phi_N)(\varepsilon)(x)$, which is specified by continuous functions $\Phi_i \in C^2(-\varepsilon_0, \varepsilon_0\ W^{1,\infty}(R^N))$ and $\Phi(0) = I$ with identity operator I. We will fix ε. Using the coordinate transformation $\Phi(\varepsilon)(x)$ for $x \in \Omega$, $x \in \partial\Omega$ and $x \in \Gamma_0$ we obtain the perturbed region $\Phi(\varepsilon)(\Omega)$ with boundary $\Phi(\varepsilon)(\partial\Omega)$ and the perturbed cut $\Gamma_\varepsilon = \Phi(\varepsilon)(\Gamma_0)$ respectively. We will define the perturbed region with the cut as $\Omega_\varepsilon = \Phi(\varepsilon)(\Omega)\backslash\overline{\Gamma}_\varepsilon$. According to the existing smoothness of the function Φ, the following expansion in series with respect to ε holds

$$\Phi(\varepsilon) = I + \varepsilon V + o(\varepsilon) \quad \text{in} \quad \mathbf{R}^N \tag{1.3}$$

where we have denoted the quantities $\partial\Phi/\partial\varepsilon$ when $\varepsilon = 0$ by the vector $V = (V_1, \ldots, V_N)$. It then follows from formula (1.3) that the Jacobian of this transformation allows of the representation

$$J(\varepsilon) \equiv |\partial\Phi/\partial x|(\varepsilon) = 1 + \varepsilon \operatorname{div} V + o(\varepsilon) \quad \text{almost everywhere in} \quad \mathbf{R}^N \tag{1.4}$$

and hence is strictly positive for fairly small ε. Consequently, the coordinate transformation

$$y_i = \Phi_i(\varepsilon)(x), \quad x = (x_1, \ldots, x_N) \in \Omega_0, \quad y = (y_1, \ldots, y_N) \in \Omega_\varepsilon \tag{1.5}$$

specifies a one-to-one correspondence between the regions Ω_0 and Ω_ε. Here the inverse transformation $x = \Phi^{-1}(\varepsilon)(y)$ exists, where $\Phi^{-1} = (\Phi_1^{-1}, \ldots, \Phi_N^{-1})$ with functions $\Phi_i^{-1}(\varepsilon) \in W^{1,\infty}(\mathbf{R}^N)$ and $\Phi^{-1}(\varepsilon)(\overline{\Omega}_\varepsilon) = \overline{\Omega}_0$.
Suppose meas $\Phi(\varepsilon)(\Gamma_D) > 0$ and the following assumption holds.

Assumption 2. For each permissible ε the collection of perturbed sets $(\Phi(\varepsilon)(\Omega), \Gamma_\varepsilon, \Omega(\varepsilon)(\Gamma_D))$, like the set $(\Omega, \Gamma_0, \Gamma_D)$, satisfies conditions *a–c* in Assumption 1.
We introduce the Sobolev space

$$\tilde{H}^1(\Omega_\varepsilon) = \{v = (v_1, \ldots, v_N)\colon v_i \in H^1(\Omega_\varepsilon),\ v_i = 0 \quad \text{almost everywhere on} \quad \Phi(\varepsilon)(\Gamma_D)\}$$

According to the one-to-one representation of regions (1.5), the differentiability of $\Phi(\varepsilon)$ and assumptions made, representation (1.5) also gives a one-to-one correspondence between these spaces $\tilde{H}^1(\Omega_0)$ and $\tilde{H}^1(\Omega_\varepsilon)$. From the inclusion $v \in \tilde{H}^1(\Omega_\varepsilon)$ it follows that $v \circ \Phi(\varepsilon) \in \tilde{H}^1(\Omega_0)$, and conversely, $v \in \tilde{H}^1(\Omega_0)$ implies $v \circ \Phi^{-1}(\varepsilon) \in \tilde{H}^1(\Omega_\varepsilon)$, where $(v \circ \Phi(\varepsilon))(x) = v(\Phi(\varepsilon)(x))$ and $(v \circ \Phi^{-1}(\varepsilon))(y) = v(\Phi^{-1}(\varepsilon)(y))$. Suppose $\nu^\varepsilon = (\nu_1^\varepsilon, \ldots, \nu_N^\varepsilon)$ is the unit vector of the normal to the perturbed cut Γ_ε. We define the set of permissible displacements in the region Ω_ε

$$K_\varepsilon = \{v = (v_1, \ldots, v_N) \in \tilde{H}^1(\Omega_\varepsilon)\colon [v] \cdot \nu^\varepsilon \geq 0 \quad \text{almost everywhere on} \quad \Gamma_\varepsilon\}$$

which will be convex and closed in $\tilde{H}^1(\Omega_\varepsilon)$. In order to obtain a one-to-one correspondence between the sets K_0 and K_ε, it is sufficient for the following condition to be satisfied.

Assumption 3. The transformation of Φ and the geometry of the cuts Γ_0 and Γ_ε are such that for all permissible values of ε the following condition is satisfied

$$\nu^\varepsilon \circ \Phi(\varepsilon) = \nu \tag{1.6}$$

Condition (1.6) will be satisfied, for example, when $\nu^\varepsilon = \nu = \text{const}$ and for an arbitrary transformation of Φ or when $\nu^\varepsilon = \nu$, which depends only on $(x_1, \ldots, x_{N-1})$ and $\Phi_i = x_i$ for all $i = 1, \ldots, N-1$.
We will now formulate the problem of equilibrium in the perturbed region Ω_ε

$$\int \sigma_{ij}(u^\varepsilon)\epsilon_{ij}(v - u^\varepsilon)d\Omega_\varepsilon \geq \int f_i(v - u^\varepsilon)_i d\Omega_\varepsilon, \quad \forall v \in K_\varepsilon \tag{1.7}$$

By virtue of Assumption 2, and from the same considerations as for problem (1.1), a unique solution $u^\varepsilon \in K_\varepsilon$ of variational inequality (1.7) exists for each permissible ε. Hence, for each fixed Φ a single-parameter family of problems (1.7) can be constructed, which depend on the region perturbation parameter ε.
The original problem (1.1) is a special case of problem (1.7) when $\varepsilon = 0$.

The asymptotic form of the solution. We can apply coordinate transformation (1.5) to the functions and integrals in inequality (1.7), so as to represent it in the original region Ω_0. The use of inverse functional matrix $\Psi = (\partial\Phi/\partial x)^{-1}$ gives the transformation of the derivatives $\partial/\partial y_i = \Psi_{ki}\partial/\partial x_k$ and the transformed strain tensor

$$E_{ij}(\Psi;\, v) = 1/2(v_{i,\,k}\Psi_{kj} + v_{j,\,k}\Psi_{ki})$$

Consequently, by Assumption 3 problem (1.7) can be rewritten in the equivalent form

$$\begin{aligned}&\int J(\varepsilon)(c_{ijkl}\circ\Phi(\varepsilon))E_{kl}(\Psi(\varepsilon);\, u^{\varepsilon}\circ\Phi(\varepsilon))E_{ij}(\Psi(\varepsilon);\, v - u^{\varepsilon}\circ\Phi(\varepsilon))d\Omega_0 \geq \\ &\geq \int J(\varepsilon)(f_i\circ\Phi(\varepsilon))(v - u^{\varepsilon}\circ\Phi(\varepsilon))_i d\Omega_0, \quad \forall v \in K_0\end{aligned} \qquad (1.8)$$

Hence, we have proved the following result.

Theorem 1. For sufficiently small ε the solution u^{ε} of the perturbed problem (1.7), represented in the original region Ω_0 using transformation (1.5), is the unique solution $u^{\varepsilon}\circ\Phi(\varepsilon) \in K_0$ of variational inequality (1.8).

Using the existing smoothness of Φf, $\{c_{ijkl}\}$, we can expand the operators in problem (1.8) in series in ε. In fact, it follows from formulae (1.3) and (1.4) that

$$\Psi(\varepsilon) \equiv (\partial\Phi/\partial x)^{-1}(\varepsilon) = I - \varepsilon\partial V/\partial x + o(\varepsilon) \quad \text{almost everywhere in} \quad \Omega_0$$

(I is the identity matrix). Hence, the following representation of the transformed strain tensor holds

$$E_{ij}(\Psi(\varepsilon);\, v) = \epsilon_{ij}(v) - \varepsilon E_{ij}(\partial V/\partial x;\, v) + o(\varepsilon)r_1(v) \quad \text{almost everywhere in} \quad \Omega_0 \qquad (1.9)$$

with a certain continuous form r_1. Here and henceforth we denote the residual terms in the expansions by r.

The expansion of the coefficients of elasticity in the form

$$c_{ijkl}\circ\Phi(\varepsilon) = c_{ijkl} + \varepsilon(V\nabla c_{ijkl}) + o(\varepsilon) \quad \text{almost everywhere in} \quad \Omega_0 \qquad (1.10)$$

also follows from formula (1.3) and the similar formula for the load functions

$$f_i\circ\Phi(\varepsilon) = f_i + \varepsilon(V\nabla f_i) + o(\varepsilon) \quad \text{almost everywhere in} \quad \Omega_0 \qquad (1.11)$$

Hence, substituting expressions (1.4) and (1.9)–(1.11) into (1.8) we obtain the asymptotic expansion of the operator on the left-hand side of this inequality in the form

$$\begin{aligned}&\int J(\varepsilon)(c_{ijkl}\circ\Phi(\varepsilon))E_{kl}(\Psi(\varepsilon);\, u)E_{ij}(\Psi(\varepsilon);\, v)d\Omega_0 = \\ &= \int(\sigma_{ij}(u)\epsilon_{ij}(v) + \varepsilon A_1(V;\, u,\, v) + o(\varepsilon)r_2(u,\, v))d\Omega_0\end{aligned} \qquad (1.12)$$

with the form

$$A_1(V;\, u,\, v) = \operatorname{div}(Vc_{ijkl})\epsilon_{kl}(u)\epsilon_{ij}(v) - \sigma_{ij}(u)E_{ij}(\partial V/\partial x;\, v) - \sigma_{ij}(v)E_{ij}(\partial V/\partial x;\, u) \qquad (1.13)$$

which is bilinear and symmetrical with respect to u and v, and the representation of the operator on the right-hand side of inequality (1.8) in the form

$$\int J(\varepsilon)(f_i\circ\Phi(\varepsilon))v_i d\Omega_0 = \int(f_i v_i + \varepsilon\operatorname{div}(Vf_i)v_i + o(\varepsilon)r_3(v))d\Omega_0 \qquad (1.14)$$

with certain continuous forms of r_2 and r_3.

The set K_0 is a cone in space $\tilde{H}^1(\Omega_0)$, and hence we choose $v = 0$ and $v = 2(u^{\varepsilon}\circ\Phi(\varepsilon))$ as the test functions in inequality (1.8), which leads to the equation

$$\begin{aligned}&\int J(\varepsilon)(c_{ijkl}\circ\Phi(\varepsilon))E_{kl}(\Psi(\varepsilon);\, u^{\varepsilon}\circ\Phi(\varepsilon))E_{ij}(\Psi(\varepsilon);\, u^{\varepsilon}\circ\Phi(\varepsilon))d\Omega_0 = \\ &= \int J(\varepsilon)(f_i\circ\Phi(\varepsilon))(u^{\varepsilon}\circ\Phi(\varepsilon))_i d\Omega_0\end{aligned}$$

The use of expansion (1.12) and (1.14) with respect to ε here gives the representation

$$\begin{aligned}\int\sigma_{ij}(u^{\varepsilon}\circ\Phi(\varepsilon))\epsilon_{ij}(u^{\varepsilon}\circ\Phi(\varepsilon))d\Omega_0 &= \int\{f_i(u^{\varepsilon}\circ\Phi(\varepsilon))_i+\varepsilon(\operatorname{div}(Vf_i)(u^{\varepsilon}\circ\Phi(\varepsilon))_i-\\ &-A_1(V;u^{\varepsilon}\circ\Phi(\varepsilon),u^{\varepsilon}\circ\Phi(\varepsilon)))+o(\varepsilon)r_4(u^{\varepsilon}\circ\Phi(\varepsilon),u^{\varepsilon}\circ\Phi(\varepsilon))\}d\Omega_0\end{aligned}\tag{1.15}$$

with a certain continuous form of r_4. Consequently, after using the Korn and Hölder inequalities, Eq. (1.15) leads to the uniform limit

$$\|u^{\varepsilon}\circ\Phi(\varepsilon)\|_{\tilde{H}_1(\Omega_0)}\le c_1\tag{1.16}$$

for sufficiently small ε.

We will now take $\upsilon = u^{\varepsilon}\circ\Phi(\varepsilon)$ in (1.1) and $\upsilon = u^0$ in (1.8) and add these inequalities. Again, taking relations (1.12) and (1.14) into account, we derive the representation

$$\begin{aligned}\int\sigma_{ij}(u^{\varepsilon}\circ\Phi(\varepsilon)-u^0)\epsilon_{ij}(u^{\varepsilon}\circ\Phi(\varepsilon)-u^0)d\Omega_0 &\le \int\{\varepsilon(\operatorname{div}(Vf_i)(u^{\varepsilon}\circ\Phi(\varepsilon)-u^0)_i-\\ &-A_1(V;u^{\varepsilon}\circ\Phi(\varepsilon),u^{\varepsilon}\circ\Phi(\varepsilon)-u^0))+o(\varepsilon)r_4(u^{\varepsilon}\circ\Phi(\varepsilon),u^{\varepsilon}\circ\Phi(\varepsilon)-u^0)\}d\Omega_0\end{aligned}$$

Similarly, using the Korn and Hölder inequalities here, taking into account limit (1.16) and the continuity of the form A_1, we obtain the estimate

$$\|u^{\varepsilon}\circ\Phi(\varepsilon)-u^0\|_{\tilde{H}^1(\Omega_0)}\le c_2\varepsilon\tag{1.17}$$

In particular, inequality (1.17) denotes that

$$u^{\varepsilon}\circ\Phi(\varepsilon)\to u^0 \text{ strongly in } \tilde{H}^1(\Omega_0) \quad\text{as}\quad \varepsilon\to 0\tag{1.18}$$

It follows from inequality (1.17), divided by ε, that the weak limit $\dot{u}(\Phi)\in\tilde{H}^1(\Omega_0)$ exists in a certain subsequence of ε_n, i.e.

$$\varepsilon_n^{-1}(u^{\varepsilon_n}\circ\Phi(\varepsilon_n)-u^0)\to\dot{u}(\Phi) \text{ weakly in } \tilde{H}^1(\Omega_0) \quad\text{as}\quad \varepsilon_n\to 0\tag{1.19}$$

This limit can be non-unique.

According to the definition in [10], this function $\dot{u}(\Phi)$ is called a weak material derivative of the solution, which can also be interpreted as the total derivative of the perturbed solution $u^{\varepsilon}\circ\Phi(\varepsilon)$ with respect to the parameter ε.

We will investigate its properties. First, we will denote by $C(u^0)\subseteq\Gamma_0$ the set of points on the cut, in which the equation $[u_0]\cdot\nu = 0$ for the solution u^0 of problem (1.1) is satisfied. Since, by virtue of condition (1.6) we have $[u^{\varepsilon}\circ\Phi(\varepsilon)]\cdot\nu\geqslant 0$ on Γ_0, on the basis of the compactness principle it follows from the convergence of (1.19) that $[\dot{u}(\Phi)]\cdot\nu\geqslant 0$ on $C(u^0)$. Second, we will use $\upsilon = u^{\varepsilon_n}\circ\Phi(\varepsilon_n)$ as the test function in inequality (1.1), divided by ε_n, and take the limit as $\varepsilon_n\to 0$ by virtue of the convergences of (1.18) and (1.19). We finally obtain

$$\int\sigma_{ij}(u^0)\epsilon_{ij}(\dot{u}(\Phi))d\Omega_0\ge\int f_i\dot{u}_i(\Phi)d\Omega_0\tag{1.20}$$

From inequality (1.8) for the subsequence of functions $u^{\varepsilon_n}\circ\Phi(\varepsilon_n)$ with test function $\upsilon = u^0$ we have, by virtue of expansions (1.12) and (1.14)

$$\begin{aligned}\int\sigma_{ij}(u^{\varepsilon_n}\circ\Phi(\varepsilon_n))\epsilon_{ij}(u^0-u^{\varepsilon_n}\circ\Phi(\varepsilon_n))d\Omega_0 &\ge \int\{f_i(u^0-u^{\varepsilon_n}\circ\Phi(\varepsilon_n))_i+\\ &+\varepsilon_n(\operatorname{div}(Vf_i)(u^0-u^{\varepsilon_n}\circ\Phi(\varepsilon_n))_i-A_1(V;u^{\varepsilon_n}\circ\Phi(\varepsilon_n),u^0-u^{\varepsilon_n}\circ\Phi(\varepsilon_n)))+\\ &+o(\varepsilon_n)r_4(u^{\varepsilon_n}\circ\Phi(\varepsilon_n),u^0-u^{\varepsilon_n}\circ\Phi(\varepsilon_n))\}d\Omega_0\end{aligned}$$

As before, we divide this inequality by ε_n and take the limit as $\varepsilon_n\to 0$ on the basis of relations (1.18) and (1.19). This gives

$$-\int\sigma_{ij}(u^0)\epsilon_{ij}(\dot{u}(\Phi))d\Omega_0\ge-\int f_i\dot{u}_i(\Phi)d\Omega_0\tag{1.21}$$

Inequalities (1.20) and (1.21) lead to the equality

$$\int \sigma_{ij}(u^0)\epsilon_{ij}(\dot{u}(\Phi))d\Omega_0 = \int f_i \dot{u}_i(\Phi)d\Omega_0 \tag{1.22}$$

We will now determine the closed convex set

$$Z(u^0) = \{v = (v_1, \ldots, v_N) \in \tilde{H}^1(\Omega_0) : [v] \cdot \nu \geq 0 \text{ on } C(u^0),$$
$$\int \sigma_{ij}(u^0)\epsilon_{ij}(v)d\Omega_0 = \int f_i v_i d\Omega_0\}$$

which is a hyperplane in the space $\tilde{H}^1(\Omega_0)$, tangent to the cone K_0 at the point u^0. This is not empty, for example, $\pm u^0 \in Z(u^0)$. Taking identity (1.22) and the previous discussions into account, we can prove the following result.

Theorem 2. Under the conditions of Theorem 1, a weak material derivative of the solution $\dot{u}(\Phi) \in Z(u^0)$ exists in the sense of the convergence of (1.19).

We can derive one more important relation characterizing $\dot{u}(\Phi)$. Problem (1.1) gives the obvious identity

$$\int \sigma_{ij}(u^0)\epsilon_{ij}(u^0)d\Omega_0 = \int f_i u_i^0 d\Omega_0$$

We subtract this from equality (1.15) with $\varepsilon = \varepsilon_n$, divide the relation obtained by ε_n and take the limit as $\varepsilon_n \to 0$ by virtue of (1.18) and (1.19). We finally obtain

$$2\int \sigma_{ij}(u^0)\epsilon_{ij}(\dot{u}(\Phi))d\Omega_0 = \int (f_i \dot{u}_i(\Phi) + \operatorname{div}(Vf_i)u_i^0 - A_1(V; u^0, u^0))d\Omega_0$$

which, taking equality (1.22) into account, gives the required result in the form of the following lemma.

Lemma 1. The following orthogonality conditions are satisfied for the material derivative $\dot{u}(\Phi)$ of Theorem 2

$$\int \sigma_{ij}(u^0)\epsilon_{ij}(\dot{u}(\Phi))d\Omega_0 = \int f_i \dot{u}_i(\Phi)d\Omega_0 = \int (\operatorname{div}(Vf_i)u_i^0 - A_1(V; u^0, u^0))d\Omega_0 \tag{1.23}$$

Note that the unique solution $U \in Z(u^0)$ of the variational inequality

$$\int \sigma_{ij}(U)\epsilon_{ij}(v - U)d\Omega_0 \geq \int (\operatorname{div}(Vf_i)(v - U)_i - A_1(V; u^0, v - U))d\Omega_0, \quad \forall v \in Z(u^0)$$

satisfies all the relations obtained for $\dot{u}(\Phi)$, but in the general case it is not possible to prove that $U = \dot{u}(\Phi)$. These functions are only identical in special cases, for example, if the condition $[u^0] \cdot \nu = 0$ is satisfied over the whole crack Γ_0 or $[U] \cdot \nu \geqslant 0$ on Γ_0, in which case the material derivative $\dot{u}(\Phi) = U$ is uniquely determined from the variational problem given above.

According to definition (1.19), the material derivative can be interpreted as the total derivative of the perturbed solution with respect to the perturbation parameter. For comparison, the Buckner weighting functions derived earlier in [16, 17] are found from the partial derivative $\lim_{\varepsilon \to 0} \varepsilon^{-1}(u^\varepsilon - u^0)$. The problems of finding a material derivative of arbitrary order from the corresponding weighting functions were constructed in [18] for the linear crack problem.

The asymptotic form of the potential energy. For a fixed perturbation Φ we can determine the potential energy function $\Pi(\Phi): (-\varepsilon_0, \varepsilon_0) \mapsto \mathbf{R}$ for problem (1.7) in the form

$$\Pi(\Phi)(\varepsilon) = \tfrac{1}{2}\int \sigma_{ij}(u^\varepsilon)\epsilon_{ij}(u^\varepsilon)d\Omega_\varepsilon - \int f_i u_i^\varepsilon d\Omega_\varepsilon$$

where $u^\varepsilon \in K_\varepsilon$ is the solution of problem (1.7). Taking into account the inequality

$$\int \sigma_{ij}(u^\varepsilon)\epsilon_{ij}(u^\varepsilon)d\Omega_\varepsilon = \int f_i u_i^\varepsilon d\Omega_\varepsilon$$

we arrive at the equivalent form

$$\Pi(\Phi)(\varepsilon) = -\tfrac{1}{2}\int f_i u_i^\varepsilon d\Omega_\varepsilon \tag{1.24}$$

In particular when $\varepsilon = 0$ we obtain the value of the potential energy for problem (1.1)

$$\Pi(\Phi)(0) = -\tfrac{1}{2}\int f_i u_i^0 d\Omega_0 \tag{1.25}$$

Under the conditions of Theorem 1, we apply coordinate transformation (1.15) to the integral in formula (1.24), which leads to the equation

$$\Pi(\Phi)(\varepsilon) = -\tfrac{1}{2}\int J(\varepsilon)(f_i \circ \Phi(\varepsilon))(u^\varepsilon \circ \Phi(\varepsilon))_i d\Omega_0 \tag{1.26}$$

We will obtain the first derivative of the potential energy function. We take the subsequence ε_n from formula (1.19), subtract the value of (1.25) from Eq. (1.26) with $\varepsilon = \varepsilon_n$, divide this relation by ε_n and take the limit as $\varepsilon_n \to 0$, by virtue of the convergences of (1.18) and (1.19) and expansions (1.4) and (1.11). We finally obtain

$$\begin{gathered}\lim_{\varepsilon_n \to 0} \varepsilon_n^{-1}(\Pi(\Phi)(\varepsilon_n) - \Pi(\Phi)(0)) = \Pi'(\Phi)(0)\\ \Pi'(\Phi)(0) = -\tfrac{1}{2}\int(\operatorname{div}(Vf_i)u_i^0 + f_i\dot{u}(\Phi))d\Omega_0\end{gathered} \tag{1.27}$$

Using Lemma 1 and identity (1.23) we can get rid of the material derivative in the second formula of (1.27) and obtain the equivalent relation

$$\Pi'(\Phi)(0) = \int(-\operatorname{div}(Vf_i)u_i^0 + \tfrac{1}{2}A_1(V; u^0, u^0))d\Omega_0 \tag{1.28}$$

Since representation (1.28) does not depend on $\dot{u}(\Phi)$, the first derivative of the function $\Pi(\Phi)$ is uniquely determined and the following formula holds

$$\Pi(\Phi)(\varepsilon) = \Pi(\Phi)(0) + \varepsilon\Pi'(\Phi)(0) + o(\varepsilon) \tag{1.29}$$

Hence we have proved the following theorem.

Theorem 3. For each permissible perturbation Φ there is a first derivative $\Pi'(\Phi)(0)$ of the potential energy function $\Pi(\Phi)$ with respect to the perturbation parameter ε for $\varepsilon = 0$, and the asymptotic formula (1.29) holds for fairly small ε.

The following lemma is an important supplement of Theorem 3.

Lemma 2. If two different perturbations $\Phi^1(\varepsilon)$ and $\Phi^2(\varepsilon)$ transfer the region with the cut Ω_0 into the same perturbed region Ω_ε for all ε, then $\Pi'(\Phi^1)(0) = \Pi'(\Phi^2)(0)$.

In fact, the solution u^ε of problem (1.7) and the energy $\Pi(\Phi)(\varepsilon)$ in representation (1.24) are determined by the perturbed region Ω_ε and is independent of the choice of the perturbation function Φ. The value of $\Pi(\Phi)(0)$ of formula (1.25) in general is independent of Φ. Hence, the unique derivative $\Pi'(\Phi)(0)$ in expansion (1.29) will also be independent in the sense of the choice of the perturbation function Φ.

Using representation (1.13), the integral in formula (1.28) can be rewritten in the form of the functional $\mathscr{L}_1$: $W^{1,\infty}(\Omega_0)^N \mapsto \mathbf{R}$, which depends on the velocity field V in the form

$$\begin{gathered}\mathscr{L}_1(V) = \int(-\operatorname{div}(Vf_i)u_i^0 + 1/2\operatorname{div}(Vc_{ijkl})\epsilon_{kl}(u^0)\epsilon_{ij}(u^0) -\\ -\sigma_{ij}(u^0)E_{ij}(\partial V/\partial x; u^0))d\Omega_0\end{gathered} \tag{1.30}$$

Integral representation (1.30) of the first derivative of the energy in the non-linear problem considered for constant coefficients of elasticity $\{c_{ijkl}\}$ were derived for the first time in [19] using the inverse coordinate transformation $\Phi^{-1}(\varepsilon) \in W^{2,\infty}(\mathbf{R}^N)^N$. The variational properties of the potential energy functional, similar to the orthogonality conditions (1.23), were used, which enabled the material derivative of the solution to be ignored.

Note also that, using the orthogonality conditions for the material derivative $\dot{u}(\Phi)$, not only the potential energy functional can be differentiated. For example, we will consider the functional of the deviation of the solution u^ε of problem (1.7) from the specified continuous finite function $w \in C_0^\infty(\Omega)$ with respect to the energy norm

$$F(\Phi)(\varepsilon) = \tfrac{1}{2}\int \sigma_{ij}(u^{\varepsilon} - w)\epsilon_{ij}(u^{\varepsilon} - w)d\Omega_{\varepsilon}$$

The function w belongs to the whole K_{ε} for fairly small ε. Substituting $\upsilon = u^0 \pm w$ into inequality (1.1) and $\upsilon = u^{\varepsilon_n} \circ \Phi(\varepsilon_n) \pm w$ into inequality (1.8) and then subtracting the relations obtained, divided by ε_n, and taking the limit as $\varepsilon_n \to 0$, we obtain the following identity (compare with conditions (1.23))

$$\int \sigma_{ij}(\dot{u}(\Phi))\epsilon_{ij}(w)d\Omega_0 = \int (\operatorname{div}(Vf_i)w_i - A_1(V; u^0, w))d\Omega_0$$

In view of the smoothness of w we have the expansion with respect to ε in series

$$w \circ \Phi(\varepsilon) = w + \varepsilon(V \cdot \nabla w) = o(\varepsilon) \text{ в } \Omega_0$$

Hence, applying transformation (1.5) to $F(\varepsilon)$ and taking into account the convergences of (1.18) and (1.19), we can calculate the following partial limit

$$\lim_{\varepsilon_n \to 0} \varepsilon_n^{-1}(F(\Phi)(\varepsilon_n) - F(\Phi)(0)) =$$

$$= \int (\tfrac{1}{2} A_1(V; u^0 - w, u^0 - w) + \sigma_{ij}(u^0 - w)\epsilon_{ij}(\dot{u}(\Phi) - V \cdot \nabla w))d\Omega_0$$

But, by virtue of conditions (1.23) and the identity obtained, we can here get rid of the material derivative $\dot{u}(\Phi)$, which gives the derivative of the functional $F(\Phi)$ with respect to ε when $\varepsilon = 0$ in the form

$$F'(\Phi)(0) = \int (\operatorname{div}(Vf_i)(u^0 - w)_i - 1/2 A_1(V; u^0 + w, u^0 - w) - \sigma_{ij}(u^0 - w)\epsilon_{ij}(V \cdot \nabla w))d\Omega_0$$

Note that the rules of differentiation with respect to ε of the functionals $F(\Phi)$ and $\Pi(\Phi)$ considered above in formula (1.27) formally correspond to the principle of differentiation of a moving volume (see [20]). But in the general case of an arbitrary functional $F(\Phi)(\varepsilon)$ this principle cannot be used here in view of the non-uniqueness of the material derivative for problem (1.1).

If the material derivative of the solution $\dot{u}(\Phi)$ is determined uniquely and the functions Φ, f and $\{c_{ijkl}\}$ are fairly smooth, a second derivative of the potential energy $\Pi''(\Phi)(0)$ exists, as was shown earlier in [15]. In the case considered here, it can be derived by analogy with the first derivative in the form

$$\Pi''(\Phi)(0) = \int (-F_2(\Phi)u^0 - \operatorname{div}(Vf_i)\dot{u}_i(\Phi) + 1/2 A_2(\Phi; u^0, u^0) + A_1(V; u^0, \dot{u}(\Phi)))d\Omega_0$$

where F_2 and A_2 are the coefficients of $\varepsilon^2/2$ in the subsequent terms of the expansions (1.14) and (1.12) respectively.

2. INVARIANT ENERGY INTEGRALS

The general form of the invariant integral. Suppose the region $D \subset \mathbf{R}^N$ with piecewise-smooth boundary ∂D is such that meas $D > 0$, $\overline{D} \subseteq \overline{\Omega}_0$ and the solution u^0 of problem (1.1) has H^2-smoothness in the region D up to the boundary. According to the previous note on the smoothness of the solution, such a region always exists. The integral from formula (1.30) can be differentiated by parts in the region D. We will denote by $q = (q_1, \dots, q_N)$ the unit vector of the outward normal to the boundary of region D. We then have the equivalent representation of functional (1.30) in the form of the sum $\mathcal{L}_1(V) = I(V) + I_1 + I_2 + I_3$ of the following integrals

$$I_1 = \int (\sigma_{ij,j}(u^0) + f_i)(V \cdot \nabla u_i^0)dD$$

$$I_2 = \int f_i(V \cdot \nabla u_i^0)d(\Omega_0 \backslash D) - \int (V \cdot \nu) f_i u_i^0 d(\partial\Omega_0 \backslash \Gamma_D) \cap (\overline{\Omega}_0 \backslash D) +$$

$$+ \tfrac{1}{2}\int (V \cdot \nabla c_{ijkl})\epsilon_{kl}(u^0)\epsilon_{ij}(u^0)d(\Omega_0 \backslash D)$$

$$I_3 = \int \sigma_{ij}(u^0)\Big(\tfrac{1}{2}\operatorname{div} V \epsilon_{ij}(u^0) - E_{ij}\Big(\frac{\partial V}{\partial x}; u^0\Big)\Big)d(\Omega_0 \backslash D)$$

and the integral over the closed boundary ∂D

$$I(V) = \int \sigma_{ij}(u^0)(\tfrac{1}{2}(V\cdot q)\epsilon_{ij}(u^0) - q_j(V\cdot \nabla u_i^0))d(\partial D) \tag{2.1}$$

Here $I_1 = 0$ by virtue of the equilibrium equations (1.2).

Our immediate aim is to represent the derivative of the energy function solely by an integral over the boundary (2.1). We will derive the sufficient conditions for which $I_2 = I_3 = 0$. The integrals in I_2 vanish when $V = 0$ or $f = 0$ and $\nabla c_{ijkl} = 0$. In order for the integral I_3 to vanish, taking into account the symmetry of the tensor $\{c_{ijkl}\}$, it is sufficient to require that the following relation should be satisfied

$$c_{ijkl}(x)\xi_{kl}(\tfrac{1}{2}\operatorname{div}V(x)\xi_{ij} - \xi_{im}V_{m,j}(x)) = 0, \quad \forall\{\xi_{ij}\}, \quad \text{for almost all} \quad x\in\Omega_0\backslash\bar{D} \tag{2.2}$$

Equation (2.2) is obviously satisfied for $V \equiv$ const. Moreover, when $N = 2$ the vector $V = x$ gives

$$\tfrac{1}{2}\operatorname{div}V = 1, \quad V_{m,j} = \delta_{mj}, \quad \tfrac{1}{2}\operatorname{div}V\xi_{ij} - \xi_{im}V_{m,j} = \xi_{ij} - \xi_{ij} = 0, \quad i,j = 1,2$$

Also, when $N = 2$ the vector $V = (-x_2, x_1)$ gives the antisymmetric matrix $V_{m,j} = V_{j,m}$ when $m \neq j$ and $V_{j,j} = 0$ $(m,j = 1,2)$, substitution of which into relation (2.2) leads formally to the condition connecting the coefficients of elasticity,

$$c_{1111} = c_{2222} = c_{1212} = -c_{1122}, \quad c_{1211} = c_{1222} = 0$$

This condition is not satisfied even for the isotropic case. When $N = 3$ one cannot construct examples of the non-constant vector $V \equiv$ const, for which relation (2.2) would be satisfied.

We will sum up these discussions in the following theorem.

Theorem 4. Suppose, under the conditions of Theorem 3, the subregions $D \subseteq \Omega_0$ with fairly smooth boundary ∂D satisfy the following assumptions: (a) the solution u^0 of problem (1.1) is from the class H^2 in $\bar{D}$, (b) in the region $\bar{\Omega}_0\backslash D$ the conditions $V = 0$ or $f = 0$, $\nabla c_{ijkl} = 0$ are satisfied, and (c) the functions V and c_{ijkl} are such that equality (2.2) holds almost everywhere in $\Omega_0\backslash\bar{D}$. Then the first derivative of the potential energy $\Pi'(\Phi)(0)$ can be represented by an invariant integral $I(V)$ of the form (2.1) along the boundaries ∂D.

In the following sections, using formula (2.1), we will construct invariant integrals for specific examples of perturbations of the region and the geometry of the cut.

Perturbation of the whole cut. We will choose a shearing function $\eta \in W^{1,\infty}(\mathbf{R}^N)$, which is finite in the region Ω and $\eta \equiv 1$ in a certain neighbourhood $\bar{\mathcal{O}}(\Gamma_0) \subset \mathbf{R}^N$ of the whole cut. Here we assume that $\Gamma_0 \subset \bar{\mathcal{O}}(\Gamma_0) \subset \operatorname{supp}\eta \subset \Omega$. Suppose the cut is plane, namely, it lies in the $x\cdot\nu = a$, $a =$ const plane. For the chosen vector $p = (p_1, \ldots, p_N)$ we consider the shift of the cut in the direction p using the perturbation $\Phi(\varepsilon) = I + \varepsilon p\eta$. Then the coordinate transformation (1.5) gives a perturbed cut Γ_ε which lies in the $(y - \varepsilon p)\cdot\nu = a$, the region $\Phi(\varepsilon)(\Omega) = \Omega$, by virtue of the fact that η is finite, and the perturbed region with the cut $\Omega_\varepsilon = \Omega\backslash\bar{\Gamma}_\varepsilon$. If Assumption 1 holds, Assumption 2 will then also be satisfied for fairly small ε. It is also obvious that condition (1.6) in Assumption 3 is satisfied by virtue of the constant vector of the normal $\nu^\varepsilon = \nu$.

We can use Theorem 4 with $\bar{D} = \operatorname{supp}\eta\backslash\mathcal{O}(\Gamma_0)$, if we additionally require that $f \equiv 0$ and $\nabla c_{ijkl} \equiv 0$ in the neighbourhood $\bar{\mathcal{O}}(\Gamma_0)$ of the cut.

In fact, the solution u^0 of problem (1.1) possesses additional local H^2-smoothness outside the neighbourhood of the edge of the cut γ_0, and of course, in $\bar{D}$, and then condition *a* of Theorem 4 is satisfied. Outside $\operatorname{supp}\eta$ we have $V \equiv 0$, in the $\bar{\mathcal{O}}(\Gamma_0)$ of the assumption $f \equiv 0$ and $\nabla c_{ijkl} \equiv 0$, and consequently condition *b* will also be satisfied. Outside $\operatorname{supp}\eta$ the function V is equal to zero, in the neighbourhood $\mathcal{O}(\Gamma_0)$ by virtue of $\eta \equiv 1$ we also have a constant velocity field $V \equiv p$, and hence equality (2.2) and condition *c* hold.

Hence, we arrive at the invariant integral $I(p\eta)$ of the form (2.1) over the boundary ∂D, consisting of the boundary of the carrier of the shearing function $\partial D_1 = \partial(\operatorname{supp}\eta)$ and the boundary of the neighbourhood $\partial D_2 = \partial\mathcal{O}(\Gamma_0)$. But the integral over ∂D_1 is equal to zero. On the other hand, by Lemma 2 the first derivative $\Pi'(I + \varepsilon p\eta)(0)$ and of course, its representation $I(p\eta)$ are independent of the choice of the shearing function η, and of course, of ∂D_2. Hence we obtain the integral

$$I(p\eta) = \int \sigma_{ij}(u^0)\left(\frac{1}{2}(p\cdot q)\epsilon_{ij}(u^0) - q_j\frac{\partial u_i^0}{\partial p}\right)d\Xi(\Gamma_0) \tag{2.3}$$

over any sufficiently smooth closed $(N-1)$-dimensional manifold $\Xi(\Gamma_0)$ around the whole cut Γ_0 from the neighbourhood $\mathcal{O}(\Gamma_0) \subset \Omega$, where $f \equiv 0$ and $\nabla c_{ijkl} \equiv 0$, and for an arbitrary vector p.

For the linear problem without the conditions for the crack surfaces not to overlap, requirement (1.6) in Assumption 3 is removed. Hence, in the linear case, the representation of the derivative of the energy of invariant integral (2.3) holds for an arbitrary geometry of the cut, which satisfies Assumption 1, and for an arbitrary direction of the shift p. For the non-linear problem investigated here, with the conditions of no overlapping, relation (1.6) is only satisfied in special cases when a non-plane cut Γ_0 and the direction of the shift p are chosen.

For example, suppose the cut Γ_0 lies on a surface which is given by the equation $x_N = \psi(x_1, \dots, x_{N-1})$ with a fairly continuous function ψ. Then the shift $\Phi(\varepsilon) = I + \varepsilon p \eta$ in the direction $p = (0, \dots, 0, p_N)$ enables condition (1.6) to be satisfied and gives invariant integral (2.3) for such a non-plane cut.

Perturbation of the edge of the cut. Suppose a plane cut Γ_0 with edge γ_0 (a closed curve when $N = 3$ or two points when $N = 2$), lies in the plane $x \cdot \nu = a$. We will assume that a certain neighbourhood of the edge of the cut $\mathcal{O}_1(\gamma_0) \subset \mathbf{R}^N$ exists, in which $f \equiv 0$ and $\nabla c_{ijkl} \equiv 0$. Here we assume that the boundary of this neighbourhood intersects the cut, which distinguishes this construction from that given in the previous section. We will choose a shearing function $\chi \in W^{1,\infty}(\mathbf{R}^N)$ with $\chi \equiv 1$ in $\mathcal{O}_1(\gamma_0)$ and $\chi \equiv 0$ outside $\mathcal{O}_2(\gamma_0)$, where $\gamma_0 \subset \overline{\mathcal{O}}_1(\gamma_0) \subset \overline{\mathcal{O}}_2(\gamma_0) \subset \Omega$ and the boundary $\mathcal{O}_2(\gamma_0)$ also intersects the cut. For the tangential vector to the cut $\tau = (\tau_1, \dots, \tau_N)$, i.e. $\tau \cdot \nu = 0$, the use of the perturbation by a shift $\Phi(\varepsilon) = I + \varepsilon\tau\chi$ converts Γ_0 into the cut Γ_ε in the same plane $y \cdot \nu = a$ with the perturbed edge γ_ε.

In the light of the assumptions made, we can use Theorem 4 with $D = \mathcal{O}_2(\gamma_0)\backslash\overline{\mathcal{O}}_1(\gamma_0)$, using the same discussions as in the previous section. The boundary ∂D will consist of three parts

$$\partial D_1 = \partial\mathcal{O}_2(\gamma_0), \quad \partial D_2 = (\overline{\mathcal{O}}_2(\gamma_0)\backslash\mathcal{O}_1(\gamma_0)) \cap \Gamma_0, \quad \partial D_3 = \partial\mathcal{O}_1(\gamma_0)$$

Part of the integral $I(\tau\chi)$ from formula (2.1) over ∂D_1 is here equal to zero by virtue of the identity $V \equiv 0$. By Lemma 2 we have that $I(\tau\chi)$ is independent of the choice of the shearing function χ, and of course, of ∂D_2. On the other hand, the integrand in $I(\tau\chi)$ over the boundary ∂D_2 is bounded in view of the additional local smoothness of the solution u^0. Hence, we can take the limit in $I(\tau\chi)$ when $\text{meas}(\partial D_2) \to 0$. As a result, only the integral by parts of the boundary ∂D_3 remains, which again, by virtue of Lemma 2, is independent of the choice of $\mathcal{O}_1(\gamma_0)$.

Hence, we have an invariant integral of the energy for a perturbation by a shift along the plane cut in the form

$$I(\tau\chi) = \int \sigma_{ij}(u^0)\Big(\frac{1}{2}(\tau \cdot q)\epsilon_{ij}(u^0) - q_j\frac{\partial u_i^0}{\partial\tau}\Big)d\Xi(\gamma_0) \tag{2.4}$$

over an arbitrary closed fairly smooth $(N-1)$-dimensional manifold $\Xi(\gamma_0)$ around the edge of the cut γ_0 in the neighbourhood $\mathcal{O}_1(\gamma_0)$ when $f \equiv 0$ and $\nabla c_{ijkl} \equiv 0$, and in an arbitrary tangential direction τ to the cut. For example, $\Xi(\gamma_0)$ can be taken in the form of a torus around the edge of the cut in $\mathbf{R}^3$.

Note that the choice of $p = \tau$ in the previous section for a shift of the whole cut leads to the same region as for a shift of the edge of the cut. Hence, it follows from Lemma 2, that $I(\tau\eta) = I(\tau\chi)$ in formulae (2.3) and (2.4) respectively.

Suppose the edge γ_0 of a plane cut in a three-dimensional region includes a rectilinear section L. As was shown in [14] for a rectangle, in this case one can obtain an invariant integral of the form (2.4) over a closed surface around only the part $\gamma_0\backslash L$ of the edge of the cut for a perturbation by a shift along L. In this case it is necessary to use relations which hold on the cut, and the additional local smoothness of the solution along L.

Invariant integrals of the type (2.3) and (2.4) were considered in [21] for the linear problem of a crack.

Perturbation of the cut tip. Consider the case $N = 2$ of a rectilinear cut Γ_0. Suppose $\tau = (\tau_1, \tau_2)$ is the unit direction vector and $C^1 = (B\tau_1, B\tau_2)$ and $C^2 = (A\tau_1, A\tau_2)$ are the two tips of the cut. In other words, the cut Γ_0 lies on a straight line passing through the origin of coordinates. We will choose the shearing functions $\chi^1, \chi^2 \in W^{1,\infty}(\mathbf{R}^2)$ in the neighbourhood of the cut tips C^1 and C^2 respectively. We will assume that they are finite, have non-intersecting carriers $\text{meas}(\text{supp}\chi^1 \cap \text{supp}\chi^2) = 0$, and $\chi^1 \equiv 1$ in the neighbourhood $\mathcal{O}(C^1)$ and $\chi^2 \equiv 1$ in the neighbourhood $\mathcal{O}(C^2)$. Discussions, given in the previous section, in this case yield the invariant integral $I(\tau(\chi^1 + \chi^2))$ and formulae (2.4) in the form of the sum of the two integrals over the non-intersecting closed fairly smooth curves $\Xi(C^1)$ and $\Xi(C^2)$ around the tips C^1 and C^2 respectively. On the other hand, using Lemma 2, we can show that these integrals are mutually independent.

Finally, we will obtain an invariant integral for each tip of the cut. Consider one tip, which does not coincide with (0, 0), and suppose this is C^1. By formula (2.4), we have the representation of the corresponding invariant integral in the form

$$I(\tau\chi^1) = \int \sigma_{ij}(u^0)\left(\frac{1}{2}(\tau \cdot q)\epsilon_{ij}(u^0) - q_j\frac{\partial u_j^0}{\partial \tau}\right)d\Xi(C^1) \tag{2.5}$$

for $f \equiv 0$ and $\nabla c_{ijkl} \equiv 0$ in the neighbourhood $\mathcal{O}(C^1)$.

Formula (2.5) is well known in fracture mechanics for the linear crack problem as a Cherepanov–Rice path independent integral. In the classical case, the function $u^0 \in \tilde{H}^1(\Omega_0)$ in (2.5) is the solution of the linear problem (compare with (1.1))

$$\int \sigma_{ij}(u^0)\epsilon_{ij}(v)d\Omega_0 = \int f_i v_i d\Omega_0, \quad \forall v \in \tilde{H}^1(\Omega_0)$$

In the non-linear case, the same expression of integral (2.5) is obtained for the solution $u^0 \in K_0$ of non-linear problem (1.1).

On the other hand, formula (2.5) is not the only possible representation of the first derivative of the energy. We will use the perturbation by a local extension $\Phi(\varepsilon) = I + \varepsilon x B^{-1}\chi^1$. This gives the same perturbed rectilinear cut Γ_ε with tips $C_\varepsilon^1 = ((B + \varepsilon)\tau_1, (B + \varepsilon)\tau_2)$ and $C_\varepsilon^2 = C^2$ as the local shear $I + \varepsilon\tau\chi^1$, considered above. For extension, the velocity field $V = xB^{-1}\chi^1$ in the neighbourhood of the cut tip $\mathcal{O}(C^1)$, where $\chi^1 \equiv 1$, has the form $V \equiv xB^{-1}$. Therefore condition (2.2) is satisfied. Hence, with the assumptions made, when the representation of the energy derivative of $\Pi'(I + \varepsilon\tau\chi^1)(0)$ by an integral over the contour (2.5) holds, the representation $\Pi'(I + \varepsilon x B^{-1}\chi^1)(0)$ in the form of the invariant integral

$$I(xB^{-1}\chi^1) = \frac{1}{B}\int \sigma_{ij}(u^0)\left(\frac{1}{2}(x \cdot q)\epsilon_{ij}(u^0) - q_j(x \cdot \nabla u_i^0)\right)d\Xi(C^1) \tag{2.6}$$

will also hold.

By virtue of Lemma 2 the following equality is satisfied

$$\Pi'(I + \varepsilon\tau\chi^1)(0) = \Pi'(I + \varepsilon x B^{-1}\chi^1)(0)$$

Therefore the invariant integrals (2.5) and (2.6) are equal to one another, i.e. $I(\tau\chi^1) = I(xB^{-1}\chi^1)$. Invariant integrals similar to (2.6) and (2.7) were introduced previously in [22] for the linear crack problem.

Note also that the same discussions as above lead to one other invariant integral in the two-dimensional case. That is, we apply the perturbation by an extension $\Phi(\varepsilon) = I + \varepsilon x\eta$ to the whole rectilinear cut, which lies on the straight line $x \cdot \nu = 0$, with shearing function η, which was found earlier. This gives an invariant integral in the form

$$I(x\eta) = \int \sigma_{ij}(u^0)\left(\frac{1}{2}(x \cdot q)\epsilon_{ij}(u^0) - q_j(x \cdot \nabla u_i^0)\right)d\Xi(\Gamma_0) \tag{2.7}$$

along an arbitrary fairly continuous closed curve $\Xi(\Gamma_0)$ around the cut Γ_0 in the neighbourhood $\mathcal{O}(\Gamma_0)$, where $f \equiv 0$ and $\nabla c_{ijkl} \equiv 0$.

3. CONCLUSION

The asymptotic representation (1.19) of the solution of the perturbed problem with respect to a small perturbation parameter with a weak material derivative of the solution holds for the non-linear crack problem with conditions for the crack surfaces not to overlap, when condition (1.6), imposed on the crack geometry and the general form of the perturbation of the region, is satisfied.

Using the material derivative method, asymptotic expansion (1.29) of the potential energy function with respect to the perturbation parameter (the analogue of the Griffith formula) has been obtained. Here the first derivative of the energy is independent of the material derivative of the solution, while a second derivative exists if the material derivative is unique. The complete asymptotic expansion of arbitrary order of both the perturbed solution and the energy function holds in the linear crack problem.

The first derivative of the energy function allows of a general representation in the form of an invariant integral over a closed contour (2.1), if the sufficient condition (2.2), imposed on the perturbation function and the coefficients of elasticity is satisfied. The invariant integral in expression (2.1) is determined by the velocity field of the chosen perturbation. Representation (2.1), with condition (2.2), holds both for the non-linear and for the linear crack problem. The next invariant integrals in the two-dimensional and three-dimensional cases are constructed: over an arbitrary smooth contour around the whole of the cut in the form of formula (2.3) for the perturbation by a shift of a plane cut in an arbitrary direction and a non-plane cut in a direction which satisfies condition (1.6), in the form of formula (2.7) for the extension of a rectilinear cut; over the contour around the edge of a plane cut in the form of formula (2.4) for the perturbation by a shift along the cut; along a contour around the tip of a rectilinear cut in the form of formula (2.5) for the perturbation by a local shift along the cut and in the form of an equivalent formula (2.6) for the perturbation by a local extension.

The formula for the invariant energy integrals obtained for the non-linear crack problem with possible contact between the crack surfaces can be used in problems of the quasi-static growth of a crack and to optimize its shape and position in the body.

I wish to dedicate this paper to the 50th year of Professor A. N. Khludnev and to thank him for consultations and for his help with the research on this topic.

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